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Existence and multiplicity results for partially superquadratic elliptic systems

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ABSTRACT

In this paper we establish the existence and multiplicity of solutions for a class of partially superquadratic elliptic systems by using the Morse theory.

1. Introduction

In this paper we consider the following elliptic system

$$\begin{cases} -\triangle u = F_u(x,u,v), & \text{in } \Omega, \\ -\triangle v = F_v(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function. The elliptic system has extensive practical backgrounds, for example, it can be used to describe the populations of two competing species [1], and the multiplicative chemical reaction catalyzed by the catalyst grains under constant or variant temperature, a correspondence of the stable station of a dynamical system determined by the reaction–diffusion system, see [2]. In recent years, many people have been devoted to study the existence and multiplicity of solutions for elliptic systems under different assumptions of growth on the nonlinear term. In [3], Alves and De Figueiredo studied a class of sublinear and superlinear elliptic system by using the topological method. In [4], Gallouët and Herbin obtained the existence of a solution for a class of elliptic systems by using the fixed point technique. In [5], Zhang and Zhang studied a class of sublinear elliptic systems by using the minimization method. In [6], Zhao and Wang studied a class of superlinear elliptic systems by using an abstract linking theorem. In [7], Zou studied a class of asymptotically linear elliptic systems by using a fountain theorem. Some related results can be found in [8,9].

In this paper, by using the Morse theory, we consider the existence and multiplicity of solutions for (1.1) under the condition that F is partially superquadratic at infinity. The main difficulty is to determine the critical groups of the functional corresponding to (1.1) at infinity, we will use a method similar as in [10,11] to overcome it.

Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$ be the eigenvalues associated with the eigenfunctions $e_1, e_2, e_3, \ldots$ of $-\triangle$ with Dirichlet boundary condition. We make the following assumptions:

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(H1) $F(x, u, v) \in C^2(\Omega \times \mathbb{R}^2, \mathbb{R})$;
(H2) There exist $C > 0$ and $1 < p < \frac{N+2}{N-2}$ such that

$$|D^2F(x, u, v)| \le C(1 + |u| + |v|)^{p-1},$$

where D^2 denote the Hessian of F with respect to u, v;
(H3) $|F_u(x, u, v) - \lambda u| = o(|u| + |v|)$ as $|u| + |v| \to \infty$, where $\lambda < \lambda_1$;
(H4) There exists a function $G \in C^2(\mathbb{R})$, constants $2 < \theta < \frac{2N}{N-2}$ and $r > 0$ such that

$$G'(v)v \ge \theta G(v) > 0, \qquad |v| \ge r$$

and

$$|F_v(x, u, v) - G'(v)| = O(|u| + |v|), \quad |u| + |v| \to \infty;$$

(H5) $F(x, u, v) = \frac{\mu}{2}(u^2 + v^2) + G_0(x, u, v)$, where $\lambda_j < \mu < \lambda_{j+1}$ for some integer $j > 0$, $|(G_0)_u(x, u, v)| + |(G_0)_v(x, u, v)| = o(|u| + |v|)$ as $|u| + |v| \to 0$;
(H6) $F(x, -u, -v) = F(x, u, v)$ for a.e. $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$.

By (H5), $u = v = 0$ is a solution of (1.1). The purpose of this paper is to find nontrivial solutions of (1.1). Now we state the main result of this paper.

Theorem 1.1. *Assume that* (H1)–(H5) *hold, then* (1.1) *has at least one nontrivial weak solution. Moreover, if* (H6) *also holds, then* (1.1) *has infinitely many pairs of weak solutions.*

The paper is organized as follows. In Section 2 we give a simple revisit to Morse theory. The proof of Theorem 1.1 will be given in Section 3.

2. Preliminaries

Let H be a real Hilbert space and $J \in C^1(H, \mathbb{R})$ be a functional satisfying the (PS) condition. Denote by $H_q(A, B)$ the q-th singular relative homology group of the topological pair with coefficients in a field $\mathcal{F}$. Let u be an isolated critical point of J with $J(u) = c$. The group

$$C_q(J, u) := H_q(J^c, J^c \setminus \{u\}), \quad q \in \mathbb{Z}$$

is called the q-th critical group of J at u, where $J^c = \{u \in H \mid J(u) \le c\}$. Denote $K = \{u \in H \setminus J'(u) = 0\}$. Assume that K is a finite set. Take $a < \inf J(K)$. The critical groups of J at infinity are defined by

$$C_q(J, \infty) := H_q(H, J^a), \quad q \in \mathbb{Z}.$$

The following results can be found in [12].

Lemma 2.1. *Suppose J satisfies the (PS) condition. If $K = \emptyset$, then $C_q(J, \infty) \cong 0, q \in \mathbb{Z}$. If $K = \{u_0\}$, then $C_q(J, \infty) \cong C_q(J, u_0), q \in \mathbb{Z}$.*

The Morse index of the critical point u is defined by the dimension of the negative space corresponding to the spectral decomposing. If the critical point u is nondegenerate, i.e., $J''(u)$ has a bounded inverse, we have the following result.

Lemma 2.2. *Suppose that $J \in C^2(H, \mathbb{R})$ and u is a nondegenerate critical point of J with Morse index j, then $C_q(J, u) = \delta_{q,j}\mathcal{F}$.*

Theorem 2.1. *Suppose that $J \in C^2(H, \mathbb{R})$ satisfies the (PS) condition, and $K = \{u_1, \dots, u_k\}$, then*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1 + t)Q(t),$$

where $Q(t)$ is a formal series with nonnegative coefficients, $M_q = \sum_{i=0}^{k} \operatorname{rank} C_q(J, u_k)$ and $\beta_q = \operatorname{rank} C_q(J, \infty)$, $q = 0, 1, 2, \dots$.

3. Proof of the main result

Let $H_0^1(\Omega)$ be the usual Sobolev space and set $E := H_0^1(\Omega) \times H_0^1(\Omega)$. Then E is a Hilbert space with inner product given by

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \int_{\Omega} (\nabla u_1 \cdot \nabla u_2 + \nabla v_1 \cdot \nabla v_2) dx, \quad \forall (u_1, v_1),\ (u_2, v_2) \in E,$$

and norm given by

$$\|(u,v)\|^2=\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx,\quad \forall (u,v)\in E.$$

Consider the functional $J:E\to\mathbb{R}$ defined by

$$J(u,v)=\frac{1}{2}\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx-\int_\Omega F(x,u,v)dx.$$

It is well known that the critical points of $J(u,v)$ correspond to the weak solutions of (1.1), and by (H1) and (H2), $J\in C^2(E,\mathbb{R})$, see [13].

For any $s\in[0,1]$, let

$$J_s(u,v)=\frac{1}{2}\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx-s\int_\Omega F(x,u,v)dx-(1-s)\int_\Omega\left(\frac{1}{2}\lambda u^2+G(v)\right)dx. \tag{3.1}$$

Lemma 3.1. *Assume that* (H1)–(H4) *hold, then for all* $s\in[0,1]$ *we have:*

(1) *J_s satisfies the (PS) condition on E;*
(2) *There is a constant A such that J_s has no critical point (u,v) satisfying $J_s(u,v)\le A$.*

Proof. (1) Let $s_n\in[0,1]$ and $\{(u_n,v_n)\}\subset E$ be a sequence satisfying

$$J_{s_n}(u_n,v_n)\le C,\qquad \|J'_{s_n}(u_n,v_n)\|=o(\|(u_n,v_n)\|) \tag{3.2}$$

as $n\to\infty$, where C is a constant independent of n. We will show that $\{(u_n,v_n)\}$ is bounded and contains a convergent subsequence.

Let $W(x,u,v)=F(x,u,v)-\frac{1}{2}\lambda u^2-G(v)$, then by (H3) and (H4),

$$\left|\int_\Omega W(x,u,v)dx\right|=O(\|(u,v)\|_2^2),\qquad \left|\int_\Omega W_v(x,u,v)dx\right|=O(\|(u,v)\|_2), \tag{3.3}$$

$$\left|\int_\Omega W_u(x,u,v)dx\right|=o(\|(u,v)\|_2) \tag{3.4}$$

as $\|(u,v)\|_2\to\infty$, where $\|(u,v)\|_2$ is the L^2 norm of (u,v), and

$$G(v)-\frac{1}{\theta}G'(v)v\le C_1 \tag{3.5}$$

for some constant C_1.

By (3.2)–(3.5), we have

$$\begin{aligned}
C+o(\|(u_n,v_n)\|^2) &\ge J_{s_n}(u_n,v_n)-\frac{1}{\theta}\langle J'_{s_n}(u_n,v_n),(u_n,v_n)\rangle\\
&=\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_\Omega(|\nabla u_n|^2-\lambda u_n^2)dx+\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_\Omega|\nabla v_n|^2dx\\
&\quad+\int_\Omega\left(\frac{1}{\theta}G'(v_n)v_n-G(v_n)\right)dx-s_n\int_\Omega W(x,u_n,v_n)dx\\
&\quad+\frac{s_n}{\theta}\int_\Omega(W_u(x,u_n,v_n)u_n+W_v(x,u_n,v_n)v_n)dx\\
&\ge\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\frac{\lambda}{\lambda_1}\right)\|(u_n,v_n)\|^2-C_1-O(\|(u_n,v_n)\|_2^2),
\end{aligned}$$

thus there exist $M_1>0$ and $M_2>0$ such that

$$\|(u_n,v_n)\|^2\le M_1+M_2\|(u_n,v_n)\|_2^2. \tag{3.6}$$

Now we show that $\|(u_n,v_n)\|_2$ is bounded. In fact if $\|(u_n,v_n)\|_2\to\infty$. Set $(\tilde{u}_n,\tilde{v}_n)=\frac{(u_n,v_n)}{\|(u_n,v_n)\|_2}$, then by (3.6) there exists a subsequence of $(\tilde{u}_n,\tilde{v}_n)$, still denoted by $(\tilde{u}_n,\tilde{v}_n)$, and $(u_0,v_0)\in E$ such that $(\tilde{u}_n,\tilde{v}_n)$ converges to (u_0,v_0) weakly in E and strongly in $L^2(\Omega)\times L^2(\Omega)$.

It is obvious that

$$\frac{1}{\|(u_n,v_n)\|_2^2}\langle J'_{s_n}(u_n,v_n),(0,v_n)\rangle=\int_\Omega|\nabla\tilde{v}_n|^2dx-\frac{1}{\|(u_n,v_n)\|_2^2}\int_\Omega G'(v_n)v_n-\frac{s_n}{\|(u_n,v_n)\|_2^2}\int_\Omega W_v(x,u_n,v_n)v_ndx,$$

and by (3.2) and (3.3),

$$\frac{1}{\|(u_n, v_n)\|_2^2}\int_\Omega G'(v_n)v_n \le C_2 \tag{3.7}$$

for some constant $C_2 > 0$.

On the one hand, by (H4)

$$G'(v_n)v_n \ge \theta G(v_n) - M \ge C(|v_n|^\theta - 1),$$

combing with (3.7), we have

$$\|(u_n, v_n)\|_2^{\theta-2}\int_\Omega |\tilde{v}_n|^\theta \le C. \tag{3.8}$$

Since $2 < \theta < \frac{2N}{N-2}$,

$$\|(u_n, v_n)\|_2^{\theta-2} \to \infty, \qquad \int_\Omega |\tilde{v}_n|^\theta \to \int_\Omega |v_0|^\theta$$

as $n \to \infty$. Hence from (3.8) we can conclude that

$$\int_\Omega |v_0|^\theta = 0,$$

therefore $v_0 = 0$.

On the other hand, for any $\varphi \in E$

$$\frac{1}{\|(u_n, v_n)\|_2}\langle J'_{s_n}(u_n, v_n), (\varphi, 0)\rangle = \int_\Omega \nabla\tilde{u}_n \cdot \nabla\varphi dx - \lambda\int_\Omega \tilde{u}_n \cdot \varphi dx - \frac{s_n}{\|(u_n, v_n)\|_2}\int_\Omega W_u(x, u_n, v_n)\varphi dx,$$

from (3.2) and (H3), as $n \to \infty$ we get

$$\int_\Omega \nabla u_0 \cdot \nabla\varphi dx - \lambda\int_\Omega u_0 \cdot \varphi dx = 0.$$

Therefore, u_0 satisfies

$$-\triangle u_0 = \lambda u_0. \tag{3.9}$$

Since $\lambda < \lambda_1$, then $u_0 = 0$. So $(u_0, v_0) = 0$, we get a contradiction with

$$\|(u_0, v_0)\|_2 = \lim_{n\to\infty}\|(\tilde{u}_n, \tilde{v}_n)\|_2 = 1.$$

Therefore, $\{(u_n, v_n)\}$ is bounded in E. Then the standard argument [13] shows that $\{(u_n, v_n)\}$ has a convergent subsequence.

In particular, for every fixed $s \in [0, 1]$, by taking $s_n = s$, $n = 1, 2, \ldots$, and processing as above, we can prove J_s satisfies the (PS) condition.

(2) Set

$$K = \{(u, v) \in E \mid J'_s(u, v) = 0 \text{ and } J_s(u, v) \le 0 \text{ for some } s \in [0, 1]\}.$$

From (1) we know that K is compact. Let

$$A = \min_{(u,v)\in K, s\in[0,1]} J_s(u, v) - 1,$$

then $J_s(u, v)$ has no critical point (u, v) satisfying $J_s(u, v) \le A$. □

By the proof of Lemma 3.1, we can obtain the following corollary.

Corollary 3.1. *There exists a constant $\delta > 0$ such that for all $s \in [0, 1]$, $a \le A$ and (u, v) with $J_s(u, v) \le a$,*

$$\|J'_s(u, v)\| \ge \delta\|(u, v)\|.$$

For any $s \in [0, 1]$, we can define the critical groups of J_s at infinity as

$$C_q(J_s, \infty) = H_q(E, J_{s,a}), \quad q = 0, 1, 2, \ldots,$$

where $a \le A$ and $J_{s,a} = \{(u, v) \in E \mid J_s(u, v) \le a\}$.

Lemma 3.2. *Assume that* (H1)–(H4) *hold, then* $C_q(J_s, \infty)$ *is independent of* $s \in [0, 1]$, *particularly*

$$C_q(J, \infty) = C_q(J_0, \infty), \quad q = 0, 1, 2 \ldots.$$

Proof. Consider the following initial value problem of the ordinary differential equation on E:

$$\frac{d}{ds}\eta(s, u, v) = -\partial_s J_s(\eta(s, u, v))\frac{J_s'(\eta(s, u, v))}{\|J_s'(\eta(s, u, v))\|^2}, \quad \eta(0, u, v) = (u, v). \tag{3.10}$$

We claim that the solution $\eta(s, u, v)$ of (3.10) exists for $s \in [0, 1]$ for any initial value $(u, v) \in E$ satisfying $J_0(u, v) \le A$. In fact, since

$$\frac{d}{ds}J_s(\eta(s, u, v)) = \partial_s J_s(\eta(s, u, v)) + J_s'(\eta(s, u, v))\frac{d}{ds}\eta(s, u, v) = 0,$$

we have $J_s(\eta(s, u, v)) \le A$ if and only if $J_0(u, v) \le A$. By (3.3) and the Sobolev inequality,

$$|\partial_s J_s(\eta(s, u, v))| = \left|\int_\Omega W(x, \eta(s, u, v))dx\right| \le C(\|\eta(s, u, v)\|^2 + 1). \tag{3.11}$$

By (3.10), (3.11) and Corollary 3.1, we have

$$\left\|\frac{d}{ds}\eta(s, u, v)\right\| \le C\frac{\|\eta(s, u, v)\|^2 + 1}{\|J_s'(\eta(s, u, v))\|} \le \frac{C}{\delta}(\|\eta(s, u, v)\| + 1).$$

By the Gronwall inequality the solution of (3.10) exists for $s \in [0, 1]$ for any initial value $(u, v) \in E$ satisfying $J_0(u, v) \le A$, the claim is proved.

Now we can define a map $\phi : J_{0,a} \to J_{1,a}$ by

$$\phi(u, v) = \eta(1, u, v),$$

which is a homeomorphism between $J_{0,a}$ and $J_{1,a}$ for $a \le A$. Hence by the definition of the critical group at infinity and the homotopy invariance of the homology group, we know that

$$C_q(J, \infty) = H_q(E, J_{1,a}) = H_q(E, J_{0,a}) = C_q(J_0, \infty), \quad q = 0, 1, 2 \ldots. \quad \Box$$

Proof of Theorem 1.1. Let

$$\Phi_1(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx, \qquad \Phi_2(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega G(v)dx,$$

then $J_0(u, v) = \Phi_1(u) + \Phi_2(v)$. Similar to the proof of Proposition 4.4 in [10], we can get the following Künneth type formula

$$C_q(J_0, \infty) = \bigoplus_{i+j=q, i,j\ge 0} C_i(\Phi_1, \infty) \otimes C_j(\Phi_2, \infty). \tag{3.12}$$

Since $\lambda < \lambda_1$, $\Phi_1(u)$ is a nondegenerate quadratic form and the Morse index of Φ_1 at zero is 0, therefore $C_q(\Phi_1, \infty) = \delta_{q,0}\mathcal{F}$. From (H4), $\Phi_2(v)$ is a superquadratic functional, then by a similar argument as [12], we can get $\Phi_2(v)$ satisfies the (PS) condition and $C_q(\Phi_2, \infty) = 0,\ q = 0, 1, 2 \ldots$. Therefore, by (3.12) and Lemma 3.2,

$$C_q(J, \infty) = C_q(J_0, \infty) = 0, \quad q = 0, 1, 2 \ldots. \tag{3.13}$$

On the other hand, by (H5), zero is a nondegenerate critical point of the functional $J(u, v)$ with the Morse index $2j$. So by Lemma 2.2,

$$C_q(J, 0) = \delta_{q,2j}\mathcal{F}. \tag{3.14}$$

Hence $C_q(J, \infty) \ne C_q(J, 0)$ for $q = 2j$. By Lemma 2.1, J has a nontrivial critical point and therefore (1.1) has at least a nontrivial solution.

Moreover, if (H6) holds, then $J(u, v)$ is an even functional. Clearly, (u, v) is a critical point of J if and only if $(-u, -v)$ is a critical point of J, (u, v) and $(-u, -v)$ have the same Morse index derived from $J''(u, v) = J''(-u, -v)$. Suppose J has a finite number of critical points. By the symmetric Mariano–Prodi perturbation we may assume the critical points of J are all nondegenerate. Let

$$K = \{(u_1, v_1), (-u_1, -v_1), (u_2, v_2), (-u_2, -v_2), \ldots, (u_k, v_k), (-u_k, -v_k)\}$$

be the nontrivial critical set of J. Then by (3.13), (3.14) and Theorem 2.1, we have

$$t^{2j} + 2\sum_{i=1}^k t^{m_i} = (1 + t)Q(t), \tag{3.15}$$

where m_i is the Morse index of (u_i, v_i) and $(-u_i, -v_i)$. Substitute $t = -1$ into (3.15), we get $1 + 2\sum_{i=1}^{k}(-1)^{m_i} = 0$, note that the right hand side of this equality is odd, we get a contradiction. Therefore, J has infinitely many pairs of critical points and (1.1) has infinitely many pairs of weak solutions. $\square$

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